

## GEAR PAIR GENERATION WITH THE METHOD OF TRANPOSED LINES OF ACTION

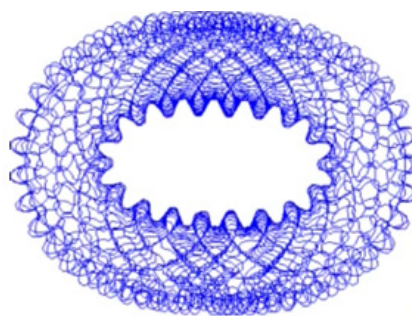
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### 1. Introduction

Noncircular gears are not wide spread in the industrial usage, which does not mean that there is no need for them. In many cases, complex mechanisms can be replaced by a single pair of noncircular gears. The reason they are not common, is that their definition and the manufacturing process is complex and time consuming. Recently various CNC technologies became available, which are suitable also to manufacture gears, like cutting with laser, wire-spark cutting, CNC milling, 3D printing. Contrary to the traditional manufacturing methods, these modern technologies need the precise geometry of the cogwheel. Creating these models is usually a slow and difficult process [Faydor and Litvin 2004], [Bendefy and Piros 2013]. Our goal was to create a different method that is simpler and enables the definition of noncircular gears with changing profile parameters.

The most typical way of calculating general gear geometry is based on the gear generating manufacturing process. After determining the pitch curves, coordinate transformations have to be carried out on the tool gear's geometry to simulate the manufacturing process. The rolling circle of the tool gear is rolled along the pitch curve. The resulting shape's envelope has to be determined that results the eventual geometry (Figure 1). One way of defining the envelope is by applying Boolean operations at each coordinate transformation.



**Figure 1. Geometry definition based on gear generating**

The method described in this article has a different approach. Instead of modelling the gear generation process, we applied the basic law of gearing [Erney 1983], [Dudley 2012], and generated the teeth geometry using the determined lines of action. These lines are escorted along the pitch curves with the help of different coordinate transformations. The connection points are registered on the gear's coordinate system, resulting the curves of the flanks. The undercut curves, top land and bottom land curves have to be determined afterwards. By merging all the curves, we can determine the gears'

geometry fast and reliably. As the curves of the gear (flanks, undercut curves, top- and bottomland curves) are calculated individually, this method also enables the application of changing profile parameters.

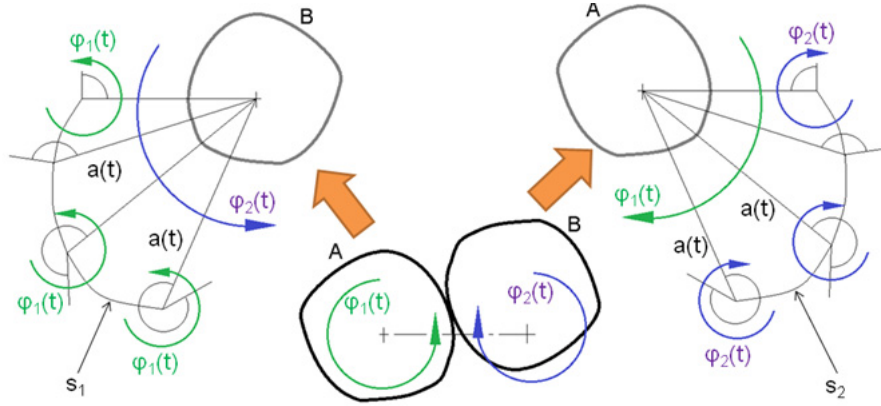
## 2. Calculating general 2-dimensional rolling (pitch) curves

As the first step of the calculation the pitch curves have to be defined. Our goal was to create curves with changing ratio and axis distance. It means that if we have a constant angular velocity on the first axis, we can create an almost arbitrary angular velocity function on the second axis as the ratio is not necessarily constant. The axis distance can also be an almost arbitrary function instead of a constant value. This enables to achieve almost arbitrary relative motion (displacement and rotation) on the gears. For defining the desired movement of the gears, the functions of  $\varphi_1(t)$  and  $\varphi_2(t)$  angular positions and  $a(t)$  axis distance have to be given. With these data the relative movements of the gears can be calculated. Knowing the relative movements, the rolling curves can be determined by applying the rigid body velocity equations (3).

As shown in Figure 2, the  $A$  and  $B$  pitch curves are to be determined. The  $S_2$  and  $S_1$  curves are the trace curves of the other coordinate systems origin and can be calculated by the relative motion (1), (2).

$$\bar{s}_1(t) = \begin{bmatrix} a(t) \sin(-\varphi_2(t)) \\ a(t) \cos(-\varphi_2(t)) \\ 0 \end{bmatrix} \quad (1)$$

$$\bar{s}_2(t) = \begin{bmatrix} a(t) \sin(-\varphi_1(t)) \\ a(t) \cos(-\varphi_1(t)) \\ 0 \end{bmatrix} \quad (2)$$



**Figure 2. The sketch used for determining the pitch curves**

The motion state of a rigid body is defined if the velocity vector of an arbitrary point and the angular velocity vector are known. This means that by applying the rigid body velocity equations (3), we can determine the velocity vector of any point. The position of this arbitrary point can be defined by an  $r_{CD}$  position vector that connects the  $C$  point (the known velocity) and the  $D$  point which's velocity is to be determined.

$$v_D = v_C + \omega \times r_{CD} \quad (3)$$

At the current connection points of the rolling curves the relative velocity is always zero. If we locate all the point with zero relative velocity, we define the rolling curves.

We apply this equation in order to find all the points that have zero relative velocity. These points are defined by the  $\bar{r}_2(t)$  vector (4).

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_{2x}(t) \\ v_{2y}(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega_2(t) \end{bmatrix} \times \begin{bmatrix} r_{2x}(t) \\ r_{2y}(t) \\ 0 \end{bmatrix} \quad (4)$$

The velocities and angular velocities can be defined by the functions of angular position and axis distance (5) (6) (Figure 2).

$$\omega_{1t}(t) = \frac{d(\varphi_1(t) - \varphi_2(t))}{dt}; \omega_{2t}(t) = \frac{d(\varphi_2(t) - \varphi_1(t))}{dt} \quad (5)$$

$$\bar{v}_{1t}(t) = \begin{bmatrix} v_{1x}(t) \\ v_{1y}(t) \\ 0 \end{bmatrix} = T_1(t) \frac{d\bar{s}_1(t)}{dt}; \bar{v}_{2t}(t) = \begin{bmatrix} v_{2x}(t) \\ v_{2y}(t) \\ 0 \end{bmatrix} = T_2(t) \frac{d\bar{s}_2(t)}{dt} \quad (6)$$

Because the points of the rolling curve are defined for a rotating coordinate system, the velocity vectors determined by the equations (6) need to be compensated by the revolution of the bodies. For this action  $T_1(t)$  and  $T_2(t)$  rotation matrixes are applied.

The final form of the pitch curves equation can be seen in (7).

$$C_1(t) = \begin{bmatrix} -\frac{v_{1y}(t)}{\omega_1(t)} \\ \frac{v_{1x}(t)}{\omega_1(t)} \\ 0 \end{bmatrix}; C_2(t) = \begin{bmatrix} -\frac{v_{2y}(t)}{\omega_2(t)} \\ \frac{v_{2x}(t)}{\omega_2(t)} \\ 0 \end{bmatrix} \quad (7)$$

### 3. Determining the curves of tooth flank

For determining the flanks the basic law of gearing or Willis's fundamental law was applied [Erney 1983], [Dudley 2012]. This law states that the line of action between the pitch point and the connection point is the common normal of the flanks curves at that specific connecting place. The flank's geometry is defined by the connecting point that goes along the line of action. From the gear's perspective the relative velocity vector of this point is always perpendicular to the line of action and tangential to the flank curve.

For the calculation a  $\xi - \eta$  coordinate system was defined that contains the line of action. This line goes across the origin of  $\xi - \eta$ . The origin point of the  $\xi - \eta$  is guided along the pitch curves, so that the  $\xi$  axis is always tangential to the curve. For this we applied different coordinate transformations. The velocity of the origin point on the pitch curve is  $v_1 = \frac{s_1}{t}$  while the connection point's velocity on the line of action is  $v_2 = \frac{s_2}{t}$ .  $s_1$  is the distance that is travelled by the origin point under  $t$  time. Under the same amount of time the  $P$  connecting point will travel  $s_2 = s_1 \cdot \cos(\alpha)$  on the line of action (Figure 3). Figure 3. Shows a simple case, where the pitch curve is a line, and the flanks are trapezoid.

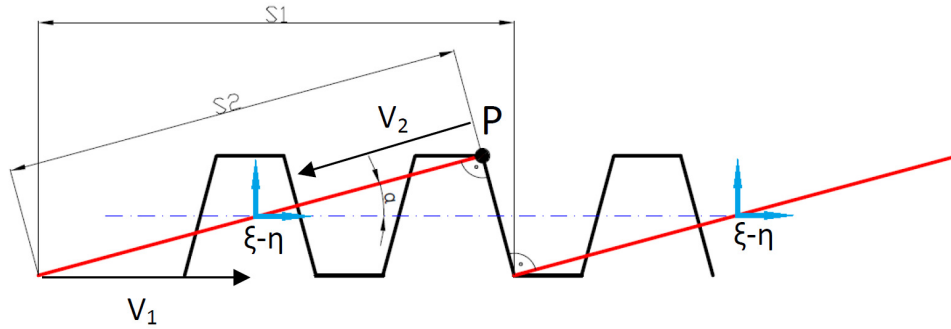


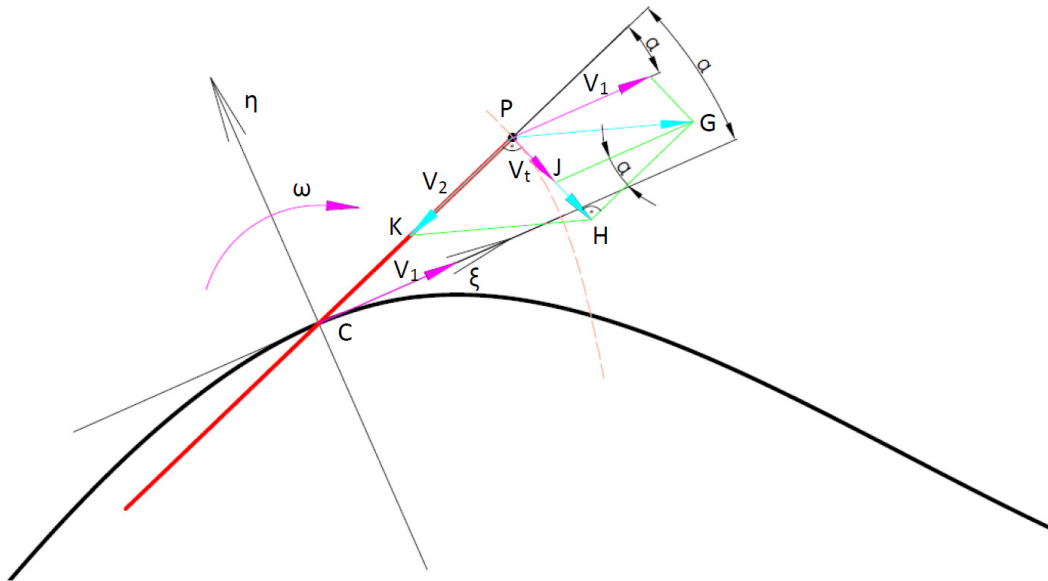
Figure 3. Lines of action and flanks at a linear pitch curve

In a general case (using the symbols of Figure 3) the velocity vector of  $\xi - \eta$  coordinate systems origin is always tangent to the pitch curve (and parallel to the  $\xi$  axis), while  $P$  connection point's velocity vector is parallel to the line of action (Figure 4). The motion state of the  $\xi - \eta$  can be defined by the  $V_1$  velocity vector of origin and the  $\omega$  angular velocity that originates from the changing of the pitch curve's slope angle. Knowing this any point's velocity vector can be determined. The  $\overline{PG}$  velocity vector of the  $P$  connection point is the sum of the vectors  $V_1$  and  $V_t$ .  $V_t$  is a velocity vector at  $P$  perpendicular to the line of action, which is caused by the  $\omega$  angular velocity (8), (Figure 4).

$$|V_t| = |\overline{PC}| \cdot \omega \quad (8)$$

$$\overline{PG} = V_1 + V_t \quad (9)$$

If  $|V_2| = |V_1| \cdot \cos(\alpha)$  the  $JHG$  angle will be perpendicular, because  $\overline{JG} = V_1$  and  $\overline{GH} = V_2$ . This equation is not influenced by the velocity vector  $V_t$ , because even if it changes, the lengths of  $|KP|$  and  $|HG|$  will be constant and equal (Figure 4).  $V_t$  has only effect on the  $PH$ ,  $HK$  and  $GP$  edges.



**Figure 4. Velocity vectors of the  $P$  connecting point**

The  $\overline{PH}$  velocity vector of  $P$  connection point is always perpendicular to the line of action or to the  $\overline{PC}$  line if the  $\frac{|V_2|}{|V_1|} = \cos(\alpha)$  is valid. It means that the generated flank curves satisfy the law of teeth connection [Erney 1983], [Dudley 2012].

A big advantage of this method is that the flank curves can be defined separately with different profile parameters. In the section below we describe the details of the calculation. First the lines of action are determined so after the corresponding flanks can be calculated.

### 3.1 Generating the rack geometry and the lines of action

For specifying the parameters we used the Table 1. In this table every two rows represent a tooth. The parameters can be set differently for each flank [Erney 1983], [Dudley 2012]. We apply this method numerically which mean that instead of a continuous curve, discrete points are calculated. The resolution of the resulting polylines can be set individually. By generating the curves, we have to make sure, that they have sections with each other, so later they can be trimmed. If the curves are not long enough, they can be extended by increasing the extension ratios.

**Table 1. Specifying the parameters**

Index	Tooth index		Profile parameters					profile			undercut			
	Gear 1.	Gear 2.	modul m, [mm]	Pressure angle $\alpha$ , [°]	height parameter $h_a^*$ , [-]	gap parameter $C^*$ , [-]	profile shift $x$ , [-]	foot	head	resolution	type	Extension	resolution	radius [mm]
1	1	1	1	10	1	0,25	0	2	2	50	1	5	100	20
2	1	2	1,2	12	1	1	0	2	2	50	1	5	100	20
3	2	2	1,4	14	1	0	0	2	2	50	1	5	100	20
4	2	3	1,6	16	1	0,25	0	2	2	50	1	5	100	20
5	3	3	1,8	18	1	0,25	0	2	2	50	2	5	100	20

First the geometry of the corresponding rack is determined using the parameters in Table 1.

The  $i$ th line of the rack is defined by the two endpoints. The coordinates of these points are

$$\begin{bmatrix} R_{x1i} \\ R_{y1i} \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} R_{x2i} \\ R_{y2i} \\ 0 \end{bmatrix}. \text{ In the expressions } i \text{ is the index of the flanks.}$$

In order to carry out the transformations on the coordinate systems containing the lines of action the end positions are required. The two end positions of the  $i$ th line of action are stored in the array  $RS_i$  (10). The parameters  $ex_{f,i}$  and  $ex_{h,i}$  are for the extension of the flank curves. The extension is essential to guaranty that the top land curves and undercut curves always cross the corresponding flank curves.

$$RS_i = [RS_{1i} \quad RS_{2i}] = \begin{bmatrix} R_{x1i} - |R_{y1i}| \tan\left(\frac{\pi}{2} - \alpha_i\right) - \frac{ex_{f,i}}{\sin(\alpha_i)} & R_{x2i} + |R_{y2i}| \tan\left(\frac{\pi}{2} - \alpha_i\right) + \frac{ex_{f,i}}{\sin(\alpha_i)} \end{bmatrix} \quad (10)$$

The endpoints of the  $i$ th line of action are stored in the array  $CL_i$  (11), (12).

$$CL_i = [CL_{1i} \quad CL_{2i}] \quad (11)$$

$$CL_{1i} = \begin{bmatrix} CL_{x1i} \\ CL_{y1i} \\ 0 \end{bmatrix}; \quad CL_{2i} = \begin{bmatrix} CL_{x2i} \\ CL_{y2i} \\ 0 \end{bmatrix} \quad (12)$$

The coordinates of the end points are determined by (13-19).

$$CL_{x1i} = |R_{y1i}| \tan\left(\frac{\pi}{2} - \alpha_i\right) + \frac{ex_{f,i}}{\sin(\alpha_i)} - ex_{f,i} \sin(\alpha_i) \quad (13)$$

$$CL_{x2i} = -|R_{y2i}| \tan\left(\frac{\pi}{2} - \alpha_i\right) - \frac{ex_{h,i}}{\sin(\alpha_i)} + ex_{h,i} \sin(\alpha_i) \quad (14)$$

$$CL_{y1i} = [CL_{y1i} \quad CL_{y2i}] \quad (15)$$

If the index  $i$  is odd the parameters  $CL_{y1i}$  and  $CL_{y2i}$  are calculated by (16), (17).

$$CL_{y1i} = |R_{y1i}| + ex_{f,i} \cos(\alpha_i) \quad (16)$$

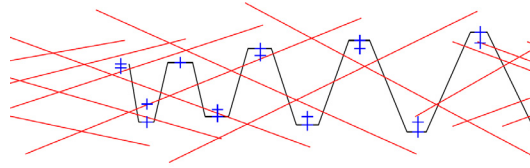
$$CL_{y2i} = -|R_{y2i}| - ex_{h,i} \cos(\alpha_i) \quad (17)$$

If the index  $i$  is even the parameters are calculated by (18), (19).

$$CL_{y1i} = -|R_{y1i}| - ex_{f,i} \cos(\alpha_i) \quad (18)$$

$$CL_{y2i} = |R_{y2i}| + ex_{h,i} \cos(\alpha_i) \quad (19)$$

The resulting rack and lines of action can be seen in Figure 5.



**Figure 5. Rack and the lines of action**

### 3.2 Generating the flank curves

The end positions and geometry of the lines of action are known so the transformations can be carried out. The pitch curves coordinates are contained by  $C_1(t)$  and  $C_2(t)$  (20) (21).

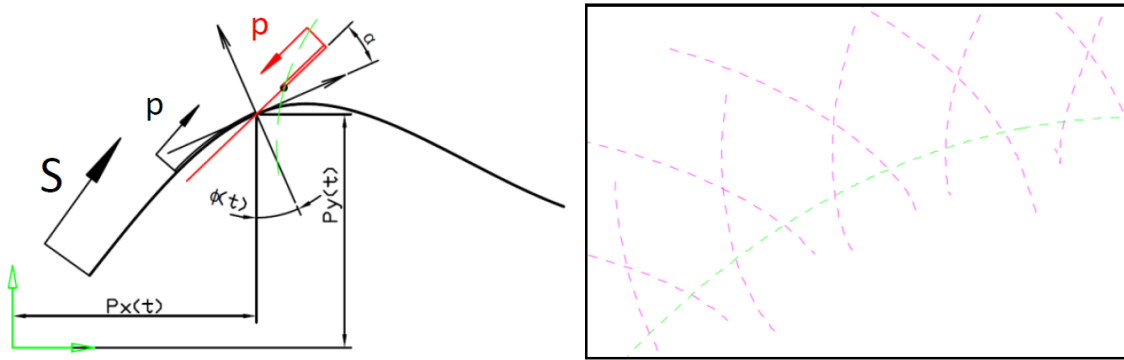
$$C_1(t) = \begin{bmatrix} C_{1x}(t) \\ C_{1y}(t) \\ 0 \end{bmatrix} \quad (20)$$

$$C_2(t) = \begin{bmatrix} C_{2x}(t) \\ C_{2y}(t) \\ 0 \end{bmatrix} \quad (21)$$

The length function of the pitch curves can be calculated by  $S(t)$  and the slope angle functions are determined by  $\phi_1(t)$  and  $\phi_2(t)$ .

The coordinate transformations have to be carried out on the  $FP_i(p)$  current point of contact (22). The parameter  $p$  changes continuously between 0 and 1 (27).

$$FP_i(p) = CL_{1i}(1 - p) + CL_{2i} \cdot p \quad (22)$$



**Figure 6. Sketch used for the coordinate transformation and the resulting curves**

The eventual equations of the flank curves (23), (24) are generated by rotating and offsetting the point of contact.

$$F_1(\hat{t}_i) = FP_i(p) \cdot T_{\phi_{1i}}(\hat{t}_i) + \bar{p}_1(\hat{t}_i) \quad (23)$$

$$F_2(\hat{t}_i) = FP_i(p) \cdot T_{\phi_{2i}}(\hat{t}_i) + \bar{p}_2(\hat{t}_i) \quad (24)$$

$T_{\phi_{1i}}(\hat{t}_i)$  and  $T_{\phi_{2i}}(\hat{t}_i)$  are rotation matrices which rotate around z axis with  $\phi_1(\hat{t}_i)$  and  $\phi_2(\hat{t}_i)$ .

In the  $t$  dependent equations a special  $\hat{t}_i$  parameter is substituted that changes continuously between  $\hat{t}_{1i}$  and  $\hat{t}_{2i}$  (25).

$$\hat{t}_i = \hat{t}_{1i} \dots \hat{t}_{2i} \quad (25)$$

The  $\hat{t}_{1i}$  and  $\hat{t}_{2i}$  parameters are determined so that if they are substituted in the length function  $S(t)$  we get the end positions of the  $i$ th line of action. (26)

$$S(\hat{t}_{1i}) = Rs_{1i}; \quad S(\hat{t}_{2i}) = Rs_{2i} \quad (26)$$

The relation between the parameters  $p$  and  $\hat{t}_i$  is represented by (27).

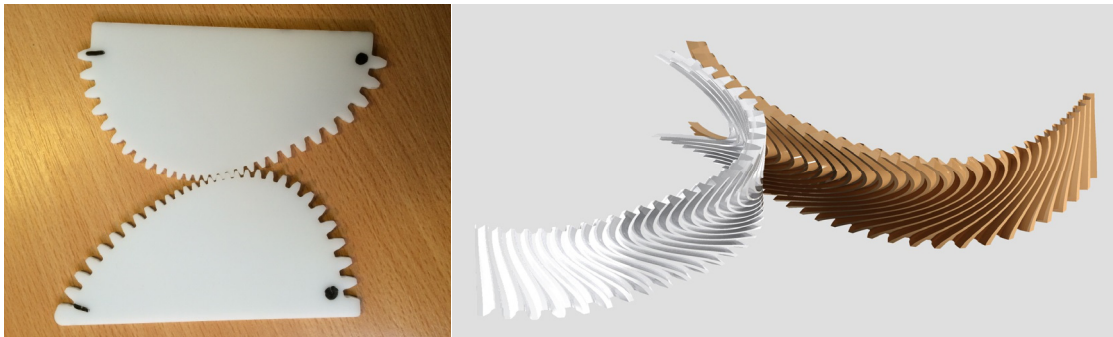
$$p = \frac{\hat{t}_i - \hat{t}_{1i}}{\hat{t}_{2i} - \hat{t}_{1i}} \quad (27)$$

The resulting curves after the transformation can be seen in Figure 6.

#### 4. Post processing the curves

After generating the flank curves, the top land, bottom land and undercut curves have to be determined. These calculations are out of the scope of this article; however it is important to mention that the undercut curves can be generated applying various methods. This is a big advantage of this calculation. In the code that we made two different undercut generating methods were worked out. First type undercut curves are generated by modelling the relative movement of a cylindrical cutting tool gear. The second type uses the relative motion of the pitch curves to each other. This method makes possible to create the most efficient undercut curves.

After determining all the curves a special algorithm is executed that merges all of them and cuts the exceeding parts. This action results the eventual 2D geometry (Figure 7).



**Figure 7. 2D and 3D gears with changing ratio, axis distance and profile parameters**

As this method is fast and robust, it enables 3D application as well. The 3D gear geometries have to be divided to parallel 2D sections. These sections can be generated separately. At a 3D parallel axis pair of gears the profile parameters can continuously change in the width direction as well. The Figure 7 shows a pair of 3D gears with changing ratio and axis distance where profile parameters change along the pitch curve and also along the width. Using the conventional method based on gear generation such gears are difficult to create.

#### 5. Conclusion

This article describes a method that can be used to generate parallel axis 2D and 3D gear geometries with changing ratio, axis distance and profile parameters. The calculation is based on the transposed lines of action. The profile parameters can change in pitch curve direction at each tooth and in width direction continuously.

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